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Multi-instantons in $\mathbb{R}^4$ and minimal surfaces in $\mathbb{R}^{2,1}$

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Abstract: It is known that self-duality equations for multi-instantons on a line in four dimensions are equivalent to minimal surface equations in three-dimensional Minkowski space. We extend this equivalence beyond the equations of motion and show that topological number, instanton moduli space and anti-self-dual solutions have representations in terms of minimal surfaces. The issue of topological charge is quite subtle because the surfaces that appear are non compact. This minimal surface/instanton correspondence allows us to define a metric on the configuration space of the gauge fields. We obtain the minimal surface representation of an instanton with arbitrary charge. The trivial vacuum and the BPST instanton as minimal surfaces are worked out in detail. BPS monopoles and the geodesics are also discussed.

Keywords: Non-perturbative Effects, Solitons Monopoles and Instantons.

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Contents

1. Introduction

Over twenty years ago Comtet [1] showed that the equations for the cylindrically symmetric multi-instantons of Yang-Mills theory given by Witten [2] correspond to minimal surface equations. He proved this equivalence at the level of the equations of motion. I extend his analysis and show that multi instantons are represented by minimal surfaces in every aspect, including the topological charge, the moduli and the anti-self-dual solutions. In particular the issue of the topological charge of instantons in terms of topological properties of the minimal surfaces is quite subtle because the minimal surfaces that appear are non compact and have infinite total curvature. So there is no well-defined notion of Euler number for these surfaces. We will show that a finite (renormalized) topological invariant, which will correspond to the topological charge of the instanton, can be defined for these surfaces. Through this construction there is also a natural way of defining a metric in the configuration space of the gauge fields.

Minimal surfaces also show up in the self-dual solutions of Einstein's equations in four dimensions. Nutku [3, 4] demonstrated that Gibbons-Hawking [5] multi gravitational instantons can be obtained from minimal surfaces in three dimensions. The correspondence follows by showing that the equations for the Ricci-flat Kähler metrics with (anti) self-dual curvature are exactly minimal surface equations in three dimensions. So the conclusion is that for every minimal surface there is a gravitational instanton.

Immediate physical relevance of the minimal surface and instanton equivalence is not clear. But one is tempted to speculate that this correspondence is reminiscent of string/gauge fields duality along the lines of [6, 7, 8]. According to Polyakov gauge invariant objects, presumably Wilson loops, are expected to have string theory representations. In this paper we will show that certain multi-instantons (objects which have gauge invariant properties) are represented by minimal surfaces which would mean a euclidean version of gauge fields/strings duality. We will not pursue this interpretation any further in this paper but leave it to future work.

The outline of the paper is as follows. In section 2 we give a review of the cylindrically symmetric multi instanton solution in four dimensions. In section 3 we describe minimal surfaces in three-dimensional Minkowski space and show that the equations describing the minimal surfaces are equivalent to the self-duality equations of the four-dimensional gauge fields. In section 3 we also define a metric on the configuration space of the gauge fields by using the metric in the minimal surface. In section 4 we find the minimal surfaces that correspond to the trivial vacuum solution and the BPST instanton and anti-instanton solutions. In section 5 by using dimensional reduction we describe charge one BPS monopole in terms of geodesics in the minimal surfaces. Section 6 consists of conclusions and discussions.

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2. Multi-instantons

In this section we will give a brief review of Witten's [2] results. Multi-instantons on a line in $\mathbb{R}^4$ are finite action solutions of the self-dual Yang-Mills fields which can be represented in the following form:

$$A^a_j(\vec{x}) = \frac{1}{r}\Big[\epsilon^a{}_{kj}\,\hat{x}^k\,(1+\varphi_2) + \delta^a{}_j\varphi_1 + (rA_1+\varphi_1)\hat{x}^a\hat{x}_j\Big]\,, \qquad A^a_0(\vec{x}) = A_0\hat{x}^a\,, \quad (2.1)$$

where all φ_i, A_i are functions of the three-dimensional radius $r\in[0,\infty]$ and the euclidean time $t\in[-\infty,\infty]$. We will work exclusively with the gauge group SU(2) so $a=(1,2,3)$. $\hat{x}^a$ are unit vectors. Setting the coupling constant to unity, the euclidean Yang-Mills action reduces to the Abelian-Higgs model in a curved space.

$$\begin{aligned} S_{\mathrm{YM}} &= \frac{1}{4}\int_{\mathbb{R}^4} d^4x F^a_{\mu\nu}F^a_{\mu\nu} \\ &= 8\pi\int_U d^2x\,\sqrt{g}\Big\{\frac{1}{2}g^{\mu\nu}D_\mu\varphi_i\,D_\nu\varphi_i + \frac{1}{8}g^{\mu\alpha}\,g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + \frac{1}{4}(1-\varphi_i^2)^2\ \Big\}\,, \end{aligned} \quad (2.2)$$

where $F_{\mu\nu}=\partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\varphi_i = \partial_\mu\varphi_i + \epsilon_{ij}A_\mu\varphi_j$. U is the upper-half plane with the Poincaré metric $g^{\mu\nu}=r^2\delta^{\mu\nu}$, yielding $ds^2=r^{-2}(dr^2+dt^2)$.[1] The (gaussian)

[1] Clearly for these cylindrically symmetric fields Yang-Mills theory in $\mathbb{R}^4$ is equivalent to a non-conformal theory in $AdS_2\times S^2$, where, AdS_2 has infinite and S^2 has a unit radius and the conformal structure is fixed by choosing the Poincaré metric on AdS_2.

scalar curvature is $K_U = -1$. From the self-duality (anti self-duality), $F^a_{\mu\nu} = \pm\tilde{F}^a_{\mu\nu}$ condition equations of motion read as

$$D_0\varphi_1 = \pm D_1\varphi_2\,, \qquad D_1\varphi_1 = \mp D_0\varphi_2\,, \qquad r^2 F_{01} = \pm(1-\varphi_1^2-\varphi_2^2)\,. \qquad (2.3)$$

The sign on the top refers to the self-dual and the lower one to anti-self-dual solutions. There is clearly a U(1) symmetry in this reduced theory. For later use let us write down how this symmetry acts on the fields.

$$\tilde{\varphi}_1 = \varphi_1\cos\theta + \varphi_2\sin\theta\,, \qquad \tilde{\varphi}_2 = -\varphi_1\sin\theta + \varphi_2\cos\theta\,, \qquad \tilde{A}_\mu = A_\mu - \partial_\mu\theta\,, \quad (2.4)$$

θ is a function of r and t. We can pick up the Lorentz gauge $\partial_\mu A_\mu = 0$ which can be solved by $A_\mu = \epsilon_{\mu\nu}\partial_\nu\psi$. Then defining $\varphi_1 = e^\psi\chi_1$ and $\varphi_2 = e^\psi\chi_2$ the first two equations in (2.3) reduce to the Cauchy-Riemann equations for the analytic function, $f(z) = \chi_1 - i\chi_2$.

$$\partial_0\chi_1 = \partial_1\chi_2\,, \qquad \partial_1\chi_1 = \partial_0\chi_2\,, \qquad (2.5)$$

where $z = r + it$. The last equation in (2.3) becomes the Liouville equation in the curved space

$$r^2\Delta\psi = |f|^2 e^{2\psi} - 1\,. \qquad (2.6)$$

The most general solution to this equation is given by

$$\psi = \log\left(\frac{2r}{(1-|g|^2)|h|}\right), \qquad g(z) = \prod_{i=1}^{k}\left(\frac{a_i - z}{\overset{*}{a}_i + z}\right),$$
$$h(z) = -i\prod_{i=1}^{k}(\overset{*}{a}_i + z)^2\,, \qquad \varphi_1 - i\varphi_2 = h\frac{dg}{dz}e^\psi\,, \qquad (2.7)$$

where $|g|^2 = \overset{*}{g}\,g$. For non-singular ψ we have $|g| = 1$ at $r = 0$, $|g| < 1$ for $r > 0$ and a_i are constants for which $\mathrm{Re}(a_i) > 0$. $k = 1$ solution corresponds to the vacuum and $k = 2$ corresponds to the BPST instanton. The locations of the zeros of dg/dz are gauge invariant and real part of a zero of this function corresponds to the instanton size and the imaginary part corresponds to the point on the t axis where the instanton is located. A careful counting shows that 2 of the parameters in (2.7) are gauge artifacts and there are $2(k-1)$ parameters for a generic solution with $k-1$ topological number.

The topological charge of the theory is

$$Q = \pm\frac{1}{8\pi^2}\int d^4x F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = \pm\frac{1}{4\pi}\int d^2x\,\epsilon_{\mu\nu}F_{\mu\nu}\,. \qquad (2.8)$$

The four-dimensional topological charge reduces to the magnetic flux in the Abelian-Higgs model. The magnetic flux is equal to the number of zeros of dg/dz multiplied by 2π giving $Q = k-1$

For later use I will write down the topological charge in the following form:

$$Q = \pm \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \int_0^{\infty} dr \frac{1}{r^2} \left(1 - \varphi_1^2 - \varphi_2^2\right) . \tag{2.9}$$

In the next section I will describe minimal surfaces in $\mathbb{R}^{2,1}$ and show that they carry all the properties of the multi-instantons that we described above.

3. Minimal surfaces

Before we give a detailed description of minimal surfaces let us mention that even a cursory look at the self-duality equations suggests that some kind of special surfaces will arise in the configuration space of this theory. We start with four functions, $A_0, A_1, \varphi_1, \varphi_2$, of r and t. The choice of gauge eliminates one of the functions, i.g. $A_0 = -A_1$, and we end up with three which describe a generic surface. Comtet [1] showed that self-duality equations (2.3) make this surface a minimal surface.

Let us recall [9] that a surface in $\mathbb{R}^3$ is a differentiable map f from a domain Σ into $\mathbb{R}^3$. Such a surface, in the non-parametric form $z = f(x, y)$, will be minimal if it satisfies

$$f_{x\,x}\,(1 + f_y^2) - 2 f_x\, f_y\, f_{x\,y} + f_{y\,y}\,(1 + f_x^2) = 0\,, \tag{3.1}$$

where f_x denotes partial differentiation. This equation describes the surfaces with vanishing mean curvature and it can be obtained by minimizing the area of the surface.

$$A = \int_\Sigma \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy\,. \tag{3.2}$$

A general conformal immersion solution to the minimal surface equation is given by the Weierstrass representation.[2]

$$f = \mathrm{Re}\left(\int \left(1 - g^2, i(1 + g^2), 2g\right) \eta \right) : \ \Sigma \to \mathbb{R}^3\,, \tag{3.3}$$

where g is a holomorphic function and η is a holomorphic one-form.

For our purposes non-parametric description, which would mean eliminating r and t from the gauge fields, is not suitable. Moreover not all the minimal surfaces can be represented in the non-parametric form. So we will use an other description which will make the instanton-minimal surface correspondence more transparent. As it is clear from the previous section the minimal surfaces that will appear are smooth sub-

[2]There is a nice spinor representation of minimal surfaces which may turn out to be quite useful for physics [10]. The spinor bundle S over Σ is a two-dimensional vector bundle which is $S = \Lambda^0 \oplus \Lambda^{(1,0)}$. So one defines the spinor $\xi = (g, \eta)$. The minimal surface equation is given by the Dirac equation for this spinor, $D(\xi) = 0$.

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domains of the upper-half-plane, U, with the Poincaré metric of constant negative curvature. Due to a theorem of Hilbert [11] we can not embed U in $\mathbb{R}^3$. So we will consider the Minkowski space $\mathbb{R}^{2,1}$ as the embedding space.

Using the notations of [12] let us consider a surface Σ in $\mathbb{R}^{2,1}$. Define orthonormal frames, $(\vec{e_\mu})$, on the surface. Here $\mu = (1,2,3)$. They satisfy $\vec{e_\mu} \cdot \vec{e_\nu} = g_{\mu\nu}$. Where $g_{\mu\nu} = \mathrm{diag}(1,1,-1)$. And we take $(\vec{e_3})$ to be orthogonal to the surface. We expect our surface to be smooth (at least twice differentiable) so we can use isothermal coordinates (u,v) for which the line element on the surface takes the form

$$ds^2 = \lambda^2(u,v)\,(du^2 + dv^2)\,. \tag{3.4}$$

There are two one-forms on the surface which we define as

$$\sigma_1 = \lambda(u,v)du\,, \qquad \sigma_2 = \lambda(u,v)dv\,. \tag{3.5}$$

A point, $\vec{x}$, which is restricted to move on the surface satisfies

$$d\vec{x} = \sigma_i \vec{e_i}\,, \tag{3.6}$$

where summation is implied for $i = (1,2)$. We need to write down how the frame moves. Let us define

$$d\vec{e_\mu} = \omega_{\mu\nu}\vec{e_\alpha}\,g^{\nu\alpha}\,, \tag{3.7}$$

where $\omega_{\mu\nu}$ are antisymmetric one forms. Using these *structure equations* one can derive the *integrability conditions* (or Gauss-Codazzi equations).

$$d\omega_{\mu\nu} - \omega_{\mu\alpha}\wedge\omega_{\beta\nu}\,g^{\alpha\beta} = 0\,, \qquad d\sigma_\mu - \omega_{\mu\nu}\wedge\sigma_\alpha\,g^{\nu\alpha} = 0\,. \tag{3.8}$$

For the surface we have $\sigma_3 = 0$. These formulae define a general surface which is not necessarily minimal. Gaussian and the mean curvature of this surface are defined in the following way:

$$K(u,v) = -\frac{1}{\lambda^2}\Delta\log\lambda\,, \qquad \frac{1}{4}\epsilon_{\mu\nu\alpha}\,\omega_{\beta\gamma}\wedge\sigma_\delta g^{\mu\beta}g^{\nu\gamma}g^{\alpha\delta} \equiv H(u,v)\sigma_1\wedge\sigma_2\,. \tag{3.9}$$

Here $\sigma_1 \wedge \sigma_2$ is the area of the surface. For the minimal surfaces $H(u,v) = 0$. So eventually we have the following sets of equations for the minimal surface:

$$\begin{aligned}
&d\sigma_1 = \tilde{\omega}\wedge\sigma_2\,, \qquad d\sigma_2 = -\tilde{\omega}\wedge\sigma_1\,, \qquad d\omega_1 = \tilde{\omega}\wedge\omega_2\,,\\
&d\omega_2 = -\tilde{\omega}\wedge\omega_1\,, \qquad d\tilde{\omega} = \omega_1\wedge\omega_2\,, \qquad \sigma_1\wedge\omega_1 + \sigma_2\wedge\omega_2 = 0\,,\\
&\sigma_1\wedge\omega_2 - \sigma_2\wedge\omega_1 = 0\,.
\end{aligned} \tag{3.10}$$

We have defined $\omega_{12} = \tilde{\omega}$, $\omega_{13} = -\omega_1$ and $\omega_{23} = -\omega_2$. These one-forms can be expressed as linear combinations of σ_1 and σ_2. Looking at the above equations and using (3.5) one obtains

$$\begin{aligned}
\omega_1 &= p(u,v)\,du + q(u,v)\,dv\,,\\
\omega_2 &= q(u,v)\,du - p(u,v)\,dv\,,\\
\tilde{\omega} &= a(u,v)\,du + b(u,v)\,dv\,.
\end{aligned} \tag{3.11}$$

Finally we have the following differential equations which describe the minimal surfaces:

$$\dot{p} - q' = a\,p + b\,q\,, \qquad \dot{a} - b' = p^2 + q^2\,, \qquad \dot{q} + p' = a\,q - b\,p\,. \tag{3.12}$$

In addition to these there are two more equations which will give us the conformal factor in the metric.

$$\dot{\lambda} = -a\lambda\,, \qquad \lambda' = b\lambda\,, \tag{3.13}$$

where $\dot{p} = \partial p/\partial v$ and $p' = \partial p/\partial u$ etc. The claim is that these equations are equivalent to the self-duality eqs. (2.3) this can be proved by making the following identifications:[3]

$$\begin{aligned} &u = r\,, \qquad v = t\,, \qquad b = \frac{1}{r} - A_o\,, \\ &p = \frac{\varphi_1}{r}\,, \qquad q = \frac{\varphi_2}{r}\,, \qquad a = -A_1\,. \end{aligned} \tag{3.14}$$

Anti-self-dual solutions can be obtained by

$$b = \frac{1}{r} + A_o\,, \qquad a = A_1\,, \qquad q = \frac{\varphi_1}{r}\,, \qquad p = \frac{\varphi_2}{r}\,. \tag{3.15}$$

With these identifications, recasting Comtet's [1] argument, we have shown that the minimal surface eqs. (3.12) are exactly equivalent to multi-instanton eqs. (2.3). We also need to find a topological invariant in the minimal surface which should correspond to the topological charge of the instanton. The gaussian curvature of the surface can be calculated to give

$$K(r,t) = \frac{1}{r^2\lambda^2}(\varphi_1^2 + \varphi_2^2)\,. \tag{3.16}$$

A naive topological invariant for the minimal surface would be the total curvature,

$$\chi = \int dA\, K(r,t) = \int_{-\infty}^{\infty} dt \int_0^{\infty} dr \frac{1}{r^2}\,(\varphi_1^2 + \varphi_2^2) \tag{3.17}$$

which clearly diverges. We will resolve this issue in the next section.

Minimal surface equations enjoy the U(1) symmetry (2.4) in the following way:

$$\begin{aligned} &\tilde{p} = p\cos\theta + q\sin\theta\,, \qquad \tilde{q} = -p\sin\theta + q\cos\theta\,, \\ &\tilde{a} = a + \theta'\,, \qquad \tilde{b} = b + \dot{\theta}\,. \end{aligned} \tag{3.18}$$

The gauge invariance of the equations involving λ can also be shown by first noting that,

$$\lambda(r,t) = \exp\left\{ -\int_{t_0}^{t} a dt + \int_{r_0}^{r} b dr \right\}. \tag{3.19}$$

[3] Equivalently one can choose the following identification; $v = r$, $u = t$, $q = \varphi_1/r$, $p = \varphi_2/r$, $a = A_0 - 1/r$ and $b = A_1$.

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We know how a and b transform from (3.18). So we find that λ transforms as

$$\lambda(\tilde{r}, t) = \lambda(r,t) \exp\left\{ -\int_{t_0}^{t} \theta' dt + \int_{r_0}^{r} \dot{\theta} dr \right\} . \tag{3.20}$$

Using this one can show that (3.13) are gauge invariant if the gauge parameter θ is a harmonic function.

We have shown that self-duality equations are in one to one correspondence with the equations that define a minimal surface. We have two more eqs. (3.13) on the minimal surface which will give us the metric on the surface. We will interpret this metric as a metric on the configuration space of the gauge fields. Using the eqs. (3.14) and the solution (2.7) one obtains the conformal factor and the metric for the self-dual solutions as

$$\lambda(r,t) = re^{-\psi(r,t)} = (1-|g|^2)|h| , \qquad ds^2 = (1-|g|^2)^2|h|^2(dr^2+dt^2) . \tag{3.21}$$

This is the metric on a minimal surface (and configuration space of gauge fields) that corresponds to a charge $k-1$ instanton. All the details of the minimal surfaces can be read off directly from the solutions of the self-duality equations. Of course the other way around is also possible by solving the minimal surface equations. For the anti-self-dual solutions we have

$$\lambda(r,t) = re^{\psi(r,t)} = \frac{r^2}{(1-|g|^2)|h|} , \qquad ds^2 = \frac{r^4}{(1-|g|^2)^2|h|^2}(dr^2+dt^2) . \tag{3.22}$$

In $\lambda(r,t)$ I have suppressed an overall constant factor.

4. BPST as a minimal surface

We start with the trivial vacuum solution, $k=1$, which is both self-dual and anti-self-dual. The general solution (2.7) yields $\varphi_2 = -1$ and $\varphi_1 = A_0 = A_1 = 0$. The metric on the field space (the minimal surface) and the gaussian curvature are,

$$ds^2_{\text{vacuum}} = r^2(dr^2+dt^2) , \qquad K_{\text{vacuum}} = \frac{1}{r^4} . \tag{4.1}$$

This is the Robertson-Walker metric. There is a singularity at the origin and the horizon is at infinity. From gauge theory side we know that the instanton number of the trivial vacuum is zero. As a minimal surface we need to look at the total gaussian curvature which turns out to be infinite. This is not a big surprise as we saw in the previous section. Our minimal surfaces are embedded in the upper-half plane with the Poincare metric, for which $K_U = -1$. The total curvature of the U-plane is infinite simply because it is non compact and has an infinite area. The lesson we learn from the vacuum solution is that we need to renormalize the total curvature of the minimal surface in order to get the correct topological charge of the gauge theory solutions. Let us write down a general formula which will be true for all instanton solutions.[4]

$$Q = \frac{1}{2\pi}\left(\int K_U dA + \int K_\Sigma dA\right). \tag{4.2}$$

Using the gaussian curvature K_Σ (3.16) of the minimal surface and $K_U = -1$ of the Poincare plane one gets the topological number for the instanton (2.9). This is a perfectly a well-defined finite number which, as we stated before, is equal to $\pm(k-1)$

Now we will find the minimal surface corresponding to the BPST [15] instanton. Choosing $k = 2$ one has

$$h(z) = -i(\overset{*}{a}_1 + z)^2 \, (\overset{*}{a}_2 + z)^2 , \qquad g(z) = \frac{(a_1 - z)\,(a_2 - z)}{(\overset{*}{a}_1 + z)\,(\overset{*}{a}_2 + z)} . \tag{4.3}$$

There are four-arbitrary parameters but a check of the zero of $dg(z)/dz$ shows that two of these parameters are redundant. The physical parameters are

$$t_0 = \frac{\mathrm{Im}(a_1\, a_2)}{\mathrm{Re}(a_1 + a_2)} , \qquad \rho^2 = -t_0^2 + \frac{\mathrm{Re}[\overset{*}{a}_1 \overset{*}{a}_2 (a_1 + a_2)]}{\mathrm{Re}(a_1 + a_2)} , \tag{4.4}$$

where t_0 is the location of the instanton on the time axis and the ρ_0 is the size of it. The metric for the self-dual solutions follows as:

$$ds^2_{\mathrm{BPST}} = r^2 \Big[r^2 + (t - t_0)^2 + \rho^2\Big]^2 (dr^2 + dt^2) . \tag{4.5}$$

It is crucial that we get the r^2 factor in front. It is the factor that cancels the infinity in the topological charge of the minimal surface (instanton) as we have seen in the vacuum case. This is again a Robertson-Walker type metric. Given the metric above one readily calculates the curvature of the minimal surface that corresponds to the BPST instanton. The curvature has a singularity at the origin. To obtain the topological charge we use (4.2). The second factor in (4.5) gives a 2π contribution to the total curvature and we obtain $Q = 1$. For the sake of completeness let us denote that

$$\psi(r, t) = -\log \frac{1}{2}\Big[r^2 + (t - t_0)^2 + \rho^2\Big]. \tag{4.6}$$

The rest of the functions (for the gauge-fields or the minimal surfaces) can be found trivially. Anti-BPST solution with topological charge -1, follows similarly.

$$ds^2_{\text{anti-BPST}} = \frac{r^2}{[r^2 + (t - t_0)^2 + \rho^2]^2} (dr^2 + dt^2) . \tag{4.7}$$

[4]This renormalization/regularization is rather standard in gravity and it was first outlined by Gibbons and Hawking [13] in the context of four-dimensional gravity. See also [14]. The reader might wonder why the boundary contributions to the topological charges are not being considered. The reason is that the boundary term would be an integral of the trace of the second fundamental form of the surface, which is the the mean curvature which vanishes for the minimal surfaces by definition.

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5. BPS monopole and geodesics

It is quite instructive to apply the results of the previous sections to the BPS monopoles. In pure Yang-Mills theory BPS monopoles can appear in a number of different ways. For example Rossi [16] showed that infinite number of instantons ($k \to \infty$) with the same instanton size and regular separation (periodic instantons) on a line appear as a BPS monopole in the limit of vanishing separation. A more direct (at the end equivalent) way is to consider all the fields to be independent of "time" t.[5] So self-duality equations become

$$A_0\varphi_2 = \partial_r\varphi_2 - A_1\varphi_1\,, \qquad \partial_r\varphi_1 + A_1\varphi_2 = A_0\varphi_1\,, \qquad -r^2\partial_r A_0 = 1 - \varphi_1^2 - \varphi_2^2\,. \tag{5.1}$$

The solution given by [18] is

$$A_0 = \frac{1}{r} - \coth(r)\,, \qquad \varphi_2 = \frac{r}{\sinh(r)}\,, \qquad A_1 = \varphi_1 = 0\,. \tag{5.2}$$

In fact one obtains a one-dimensional moduli space of solutions corresponding to the value of $A_0(\infty)$, which I have assumed to be -1 here.

After dimensional reduction the minimal surface equations become

$$q' = -bq\,, \qquad b' = -(p^2+q^2)\,, \qquad p' = -bp\,, \qquad \lambda' = b\lambda\,. \tag{5.3}$$

The solution follows as:

$$a = p = 0\,, \qquad q = \frac{1}{\sinh(r)}\,, \qquad b = \coth(r)\,, \qquad \lambda = \sinh(r)\,. \tag{5.4}$$

This is a geodesic in the minimal surface. The curvature of this geodesic is

$$K(r) = \frac{1}{\sinh^4(r)}\,. \tag{5.5}$$

Along the lines of the discussion of topological charge from the previous section we need to renormalize the total curvature of this geodesic to get the correct topological number for the BPS monopole. The geodesics in the upper half plane are given by semi-circles and the vertical lines that are orthogonal to the t-axis. In the limit of vanishing t only the vertical geodesic at the origin survive, $ds^2 = r^{-2}dr^2$. Its curvature is -1. So we need to subtract the total curvature of this vertical geodesic from the BPS geodesic. It follows that[6]

$$Q_{\mathrm{BPS}} = -\int_0^\infty dr\,K(r)\,\lambda(r)^2 + \int_0^\infty dr\,\frac{1}{r^2} = 1\,. \tag{5.6}$$

[5] We can only get a charge one monopole through this construction. Hitchin [17] gave an implicit construction of multi-monopoles from geodesics.

[6] To take care of the trivial factor of 2π in front of (3.17) one can think that we compactify the t-direction on a circle of unit radius

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6. Conclusion

Extending the analysis of Comtet [1] we have shown that cylindrically symmetric multi-instantons are equivalent to minimal surfaces in three-dimensional Minkowski space. At the level of the equations of motion this equivalence follows rather directly. The issue of topological charges was subtle and we showed that the "Euler number" of the minimal surface requires a renormalization to get the correct topological number for the corresponding instanton. One can also interpret this renormalization as adding an Einstein-Hilbert action (Euler number) to the Abelian-Higgs model action that was discussed in the first section. Gravity in two dimensions is non dynamical and we know that for our particular model we have AdS_2 with the Poincaré metric.

These instantons with topological charge n have $2n$-dimensional moduli spaces. The corresponding minimal surfaces have $2n$ moduli as it can be seen from the most general explicit solution of the minimal surface equations. The dimension of the moduli space do not have a simple expression in terms of the genus of the surface since our surfaces are not closed. Anti-self dual solutions are related to the self-dual solutions in a rather non-trivial way as can be seen from eqs. (3.21) and (3.22).

Through our construction there is a natural metric defined on the configuration space of the gauge fields which is the metric on the minimal surface. We have given two explicit examples of these. The first one is the trivial vacuum solution and the other one is the BPST instanton. We worked out the details of the minimal surfaces that correspond to these solutions.

Charge one BPS monopole/geodesics equivalence is a natural byproduct of our construction after dimensional reduction. In this case one again needs to renormalize the total curvature of the BPS geodesic to get the correct topological charge for the BPS monopole.

Gibbons-Hawking gravitational multi-instantons can also be derived from minimal surfaces as it was shown by Nutku [3]. The issue of the topological charge and the moduli in this correspondence needs to be studied in detail.

A possible further direction of research would be to understand how this correspondence fits in the picture of gauge fields and string/duality. Self-dual gauge fields is perhaps a simple system of gauge fields where one can establish with some rigor an equivalence between field theory and string theory. We leave these discussion for future work.

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References

[1] A. Comtet, *Instantons and minimal surfaces*, *Phys. Rev.* **D 18** (1978) 3890.

[2] E. Witten, *Some exact multipseudoparticle solutions of classical Yang-Mills theory*, *Phys. Rev. Lett.* **38** (1977) 121.

[3] Y. Nutku, *Gravitational instantons and minimal surfaces*, *Phys. Rev. Lett.* **77** (1996) 4702.

[4] A.N. Aliev, J. Kalayci and Y. Nutku, *General minimal surface solution for gravitational instantons*, *Phys. Rev.* **D 56** (1997) 1332.

[5] G.W. Gibbons and S.W. Hawking, *Gravitational multi-instantons*, *Phys. Lett.* **B 78** (1978) 430.

[6] A.M. Polyakov, *String theory and quark confinement*, *Nucl. Phys.* **68** *(Proc. Suppl.)* (1998) 1 [hep-th/9711002].

[7] A.M. Polyakov, *The wall of the cave*, *Int. J. Mod. Phys.* **A 14** (1999) 645 [hep-th/9809057].

[8] A.M. Polyakov and V.S. Rychkov, *Loop dynamics and AdS/CFT correspondence*, hep-th/0005173.

[9] R. Osserman, *A survey of minimal surfaces*, Dover Publications, New York 1986.

[10] R. Kusner, N. Schmitt, *The spinor representation of surfaces in space*, dg-ga/9610005.

[11] W.P. Thurston, *Three-dimensional geometry and topology*, Princeton University Press, New Jersey 1997.

[12] H. Flanders, *Differential forms with applications to physical sciences*, Academic Press, New York 1963.

[13] G.W. Gibbons and S.W. Hawking, *Action integrals and partition functions in quantum gravity*, *Phys. Rev.* **D 15** (1977) 2752.

[14] S.W. Hawking, G.T. Horowitz, *The gravitational hamiltonian, action, entropy and surface terms*, *Class. and Quant. Grav.* **13** (1996) 1487 [gr-qc/9501014].

[15] A.A. Belavin, A.M. Polyakov, A.S. Shvarts and Y.S. Tyupkin, *Pseudoparticle solutions of the Yang-Mills equations*, *Phys. Lett.* **B 59** (1975) 85.

[16] P. Rossi, *Propagation functions in the field of a monopole*, *Nucl. Phys.* **B 149** (1979) 170.

[17] N.J. Hitchin, *Monopoles and geodesics*, *Comm. Math. Phys.* **83** (1982) 579.

[18] M.K. Prasad and C.M. Sommerfield, *An exact classical solution for the't Hooft monopole and the Julia-Zee dyon*, *Phys. Rev. Lett.* **35** (1975) 760.